PROBLEMS IN REAL ANALYSIS

Problem 1

Given $f \in C([0,\infty))$ such that $f(x) \to 0$ as $x \to \infty$ show that for any $\epsilon > 0$ there is a polynomial p such that $|f(x) - e^{-x}p(x)| < \epsilon \; \forall x \in [0,\infty)$.

[See also problems 16 and 109 below]
Problem 2

If K is a compact subset of $\mathbb{R}^n$ show that the set $A = \{x \in \mathbb{R}^n : d(x,K) = 1\}$ has Lebesgue mesure 0.

Problem 3
If $f_n \to 0$ a.e. on a finite measure space $(\Omega, \mathcal{F}, \mu)$ show that there is a sequence $\{a_n\} \uparrow \infty$ such that $a_n f_n \to 0$ a.e.

Problem 4
Let $x_1, x_2 \in \mathbb{R}^2$. If $A \subset \mathbb{R}^2$ has positive Lebesgue measure show that there exists $y \in \mathbb{R}^2$ and $t \in \mathbb{R}\backslash\{0\}$ such that $y + tx_1$ and $y + tx_2$ both belong to A.

More generally if F is a finite subset of $\mathbb{R}^n$ and $m(A) > 0$ then there exists $y \in \mathbb{R}^2$ and $t \in \mathbb{R}\backslash\{0\}$ such that $y + tx$ belongs to A for all $x \in F$.
Problem 5
If A and B are subsets of $\mathbb{R}$ of positive measure show that $A + B$ contains an open interval.

Problem 6
If A is a measurable subset of $\mathbb{R}$ such that $a \in A, b \in B, a \neq b \Rightarrow \frac{a+b}{2} \notin A$ then A has measure 0.

Problem 7 [Steinhaus, 1920, Fund. Math.]

Let A be a measurable subset of $\mathbb{R}$ with positive measure. Let $x_1, x_2, ..., x_k$ be distinct real numbers. Then there exists $c \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$ such that $c + tx_i \in A$ for $1 \leq i \leq k$. [Thus, if we are given $d_1, d_2, ..., d_k \in (0,\infty)$ we can fing points in A such that distnces between them are in proportion to $d_1, d_2, ..., d_k$].

Problem 8
Let $f : [a,b] \to \mathbb{R}$. Then f is Lebesgue measurable if and only if the following condition holds: for any $\epsilon > 0$ and any measurable set $A \subset [a,b]$ with $m(A) > 0$ there is a measurable subset B of A such that $m(B) > 0$ and the oscillation of f on B is atmost ϵ.

Problem 9

There is no metric d on the set of all Borel measurable maps $: \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ pointwise if and only if $d(f_n, f) \to 0$.

Problem 10

If $(a_n, b_n) \uparrow (a, b)$ and $f \in C^\infty(\mathbb{R})$ is a polynomial on (a_n, b_n) for each n show that f is a polynomial on (a, b).

Problem 11

If $f \in C^\infty(\mathbb{R})$ and, for each $x \in \mathbb{R}$ there is an integer $n \geq 0$ such that $f^{(n)}(x) = 0$ then f is a polynomial.

Problem 12

Let (X, d) be a complete metric space and $A \subset X$. Show that there is an equivalent metric on A which makes it complete if and only if A is a G_δ in X.

Problem 13

If $\{a_n\}, \{b_n\}$ are sequences of real numbers such that $a_n \cos(nx) + b_n \sin(nx) \to 0$ as $n \to \infty$ on a set E of positive measure show that $a_n \to 0$ and $b_n \to 0$.

Problem 14

If E is a set of finite measure in $\mathbb{R}$ show that $\int\limits_E \cos^{2m}(nx - \alpha_n)dx \to m(E)\frac{1}{2\pi}\begin{pmatrix} 2m \\ m \end{pmatrix} 2^{-2m}$ as $n \to \infty$ for any positive integer m and any $\alpha_n' s \in \mathbb{R}$.

Use this to prove the following generalization of Problem 13: $\limsup |a_n \cos(nx) + b_n \sin(nx)| = \limsup [a_n^2 + b_n^2]^{1/2}$ almost everywhere if $\{(a_n, b_n)\}$ is bounded.

Problem 15

If $f : \mathbb{R}^2 \to \mathbb{R}$ is separately continuous then it is continuous on a dense set.

Problem 16

Prove or disprove: if $\phi : [0, \infty) \to \mathbb{R}$ is continuous and $\phi(x)p(x) \to 0$ as $x \to \infty$ for every polynomial p then the conclusion of Problem 1 holds with e^{-x} replaced by $\phi(x)$. [i.e. given $f \in C([0, \infty))$ such that $f(x) \to 0$ as $x \to \infty$ and $\epsilon > 0$ there is a polynomial p such that $|f(x) - \phi(x)p(x)| < \epsilon \; \forall x \in [0, \infty)$].

Problem 17

Show that any $\sigma-$ algebra on $\mathbb{N}$ is generated by a finite or countable infinite partition.

[Corollary: any measure on any $\sigma-$ algebra on $\mathbb{N}$ extends to a measure on the power set].

Problem 18

If $f : [0,\infty) \to [0,\infty)$ is continuous and if $\sum_{n=1}^{\infty} f(nx) < \infty$ for all $x \geq 0$ show that $\int_0^{\infty} f(x)dx < \infty$. If $\sum_{n=1}^{\infty} f(nx) = \infty$ for all $x \geq 0$ does it follow that $\int_0^{\infty} f(x)dx = \infty$? If $f : [0,\infty) \to [0,\infty)$ is continuous and $\int_0^{\infty} f(x)dx < \infty$ does it follow that $\sum_{n=1}^{\infty} f(n) < \infty$?

Problem 19

Let $f : [0,\infty) \to [0,\infty)$ is continuous and $f(x+y) - f(x) \to 0$ as $x \to \infty$ for each $y \in [0,\infty)$. Show that the convergence is uniform for y in compact subsets of $[0,\infty)$.

Remark1

$g_n \to 0$ uniformly on $[0,\infty)$ if and only if f is a constant.

Remark 2

Under the hypothesis of this problem, f is necessarily uniformly continuous.

Problem 20

Does there exist a non-constant bounded C^∞ function : $\mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x) \geq 0$ $\forall n \geq 0$, $\forall x \in \mathbb{R}$?

If yes, give a counter-example. If no, give a real-analytic proof (as opposed to a complex analytic proof).

Problem 21

Find a necessary and sufficient condition on a continuous function f on $[0,1]$ under which it can be approximated uniformly by polynomials with integer coefficients.

Problem 22

If $A \subset \mathbb{R}$ is measurable, $\{x_n\}$ is dense and $x_n + A = A$ $\forall n$ show that either $m(A) = 0$ or $m(A^c) = 0$.

Problem 23

Let $f : [0,1] \to \mathbb{R}$ be a function such that for every $\epsilon > 0$ there is a $\delta > 0$ with $\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon$ whenever $n \geq 1$ and $\sum_{j=1}^{n} |b_j - a_j| < \delta$. Show that f is Lipschitz.

Problem 24

Let $a_i < b_i \; \forall i \in I$. Show that $\bigcup_{i \in I} [a_i, b_i]$ can be written as $\bigcup_{n=1}^{\infty} [a_{i_n}, b_{i_n}]$ for some sequence $\{i_n\} \subset I$.

Problem 25

Let $a < b$ and $\mathcal{F}$ be a collection of closed non-denerate intervals such that $x \in [a, b]$ implies there exists $\delta > 0$ (possibly depending on x) such that every closed interval of length less than δ containing x belongs to $\mathcal{F}$. Show that there is a partition $\{t_i\}$ of $[a, b]$ such that $[t_{i-1}, t_i] \in \mathcal{F} \; \forall i$.

Problem 26

Prove that $[a, b]$ is compact using Problem 25.

Problem 27

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for each real number x there is a $\delta > 0$ with $f(y) \geq f(x) \; \forall y \in (x, x + \delta)$ and $f(y) \leq f(x) \; \forall y \in (x - \delta, x)$. Prove that f is non-decreasing.

Problem 28

Let $f : [a, b] \to \mathbb{R}$ be differentiable. Show that f is absolutely continuous if and ony if it is of bounded variation.

Problem 29

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $F(x) = \sup\{f(x + h) : 0 \leq h \leq \delta\} \in \mathbb{R} \cup \{\infty\}$. Then F has right and left limits at every point.

Problem 30

[This is related to Problem 24 above]. Let A be the union of a family of closed balls (of positive radius) in $\mathbb{R}^n$. Is A necessarily a Borel set?

Remark: it is known that an arbitrary union of closed balls (of positive radius) in $\mathbb{R}^n$ is Lebesgue measurable.

Problem 31

Prove or disprove that if p is a polynomial of degree n with leading coefficient 1 then $\{x : p(x) > 0, p'(x) > 0, ..., p^{(n)}(x) > 0\}$ is an (open) interval (which may be empty, of course).

Remark: the same argument works for $\{x : p(x) < 0, p'(x) < 0, ..., p^{(n)}(x) < 0\}$. Thus, there is no need to assume that the leading coefficient is 1.

Problem 32

Let $f \in C[0,1]$ and $0 < t_n \downarrow 0$. Suppose there is a constant $C \in (0, \infty)$ such that $|f(x + t_n) - f(x)| \leq Ct_n$ for all n and x with $0 \leq x < x + t_n \leq 1$. Show that f absolutely continous and that it is also of bounded variation.

Remark: if $\frac{f(x+t_n)-f(x)}{t_n} \to 0$ "boundedly" for some $\{t_n\} \downarrow 0$ and f is continuous then f is a constant.

Problem 32

There is a set $E \subset [0,1]$ of measure 0 such that every Riemann integrable function f on $[0,1]$ has at least one point of continuity in E.

Problem 33

If $f \in L^1(\mathbb{R})$ and $\displaystyle\int |f(x + y) - f(x)|\, dx = \circ(y)$ as $y \to 0+$ show that $f = 0$ a.e..

Problem 34

Let μ be a finite positive measure (or a complex measure) on the Borel $\sigma-$field of $\mathbb{R}$. Let $0 < c < 1$ and suppose $m(A) = c \Rightarrow \mu(A) = 0$ (where m is the Lebesgue measure). Show that $\mu = 0$.

Problem 35

Show that any $f \in C[0,1]$ can be written as $g + h$ where g and $h \in C[0,1]$ and they are both nowhere differentiable.

Remark. Any bounded measurable function on $\mathbb{R}$ is the sum of two bounded measurable functions each of which is one-to-one.

Problem 36

Construct a topological space (X, τ) and a sequence of measurable functions $\{f_n\}$ from $[0,1]$ into X such that $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists $\forall x \in [0,1]$ but f is not measurable. [Here measurability is w.r.t. the Borel $\sigma-$ fields on $[0,1]$ and X].

Remark

If X is a metric space and $\{f_n\}$ is a sequence of measurable functions $\{f_n\}$ from $[0,1]$ into X such that $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists $\forall x \in [0,1]$ then f is measurable.

Problem 37

Let H be a complex Hilbert space and $T : H \to H$ an isometry which is not onto. Show that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Problem 38

Let H be a Hilbert space and P, Q be projections on M and N respectively. Prove that $\{(PQ)^n x\}$ converges for every x. What can you say about the operator $\lim_{n\to\infty}(PQ)^n$?

Problem 39

Let M be a closed linear subspace of $L^1[0,1]$ such that $M \subset \cup\{L^p[0,1] : p > 1\}$. Show that $M \subset L^p[0,1]$ for some $p > 1$.

Problem 40

Prove or disprove: if $k \in \mathbb{N}$ and $\{p_n\}$ is a sequence of polynomials of degree not exceeding k converging pointwise to 0 on $[0,1]$ then $p_n \to 0$ uniformly.

Remark

There exist sequence of polynomials on $\mathbb{C}$ converging pointwise to a discontinuous function.

Problem 41

Let (X, d) be a metric space such that every decreasing sequence of closed sets with diameters approaching 0 has non-empty intersection. Can we conclude that (X, d) is complete?

Problem 42

Let $f : [0,1] \to \mathbb{R}$ be continuous and non-decreasing. Show that there is a sequence of polynomials $\{p_n\}$ such that $p_n \uparrow f$ uniformly on $[0,1]$ and each p_n is non-decreasing.

Problem 43

Let $f : [0,1] \to \mathbb{R}$ be continuous and one-to-one. Show that there is a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[0,1]$ and each p_n is one-to-one.

f is strictly increasing and we find a striclty increasing sequence of strictly increasing polynomials $\{\xi_n\}$ converging uniformly by the argument of Problem 42.

Problem 44

If P, Q and PQ are projections on a Hilbert space and $P \neq Q$ show that $\|P - Q\| = 1$.

Problem 45

Let p be q be polynomials with real coefficients. Show that if $\max\{p(x), q(x)\}$ is a polynomial then either $p(x) \leq q(x)\ \forall x$ or $q(x) \leq p(x)\ \forall x$. Show that the same conclusion holds if $\min\{p(x), q(x)\}$ is a polynomial.

Problem 46

Find a necessary and sufficient condition on a sequence $\{b_n\}$ of real numbers that $\sum a_n b_n$ converges whenever $\sum a_n$ converges.

Problem 47

Consider the colection of all polynomials on $[0, 1]$ with the ordering $p \leq q$ if $p(x) \leq q(x)\ \forall x$. Let p and q be any two polynomials. Show that one of the following is true:

a) $p(x) \leq q(x)\ \forall x$ or $q(x) \leq p(x)\ \forall x$

b) there is no smallest polynomial ϕ exceeding both p and q

Problem 48

Show that if T and S are commuting operators on a normed linear space that $\rho(T + S) \leq \rho(T) + \rho(S)$ where $\rho(T) = \limsup \|T^n\|^{1/n}$ (the spectral radius of T). Give examples of 2×2 matrices A and B such that $\rho(A+B) > \rho(A)+\rho(B)$.

Problem 49

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, integrable and of bounded variation. Show that $\sum\limits_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{2\pi} \sum\limits_{n=-\infty}^{\infty} \hat{f}(n)$.

Problem 50

Let $\{f_n\}$ be an orthonormal basis of $L^2([0, 2\pi])$. Show that $\sum\limits_{n=-\infty}^{\infty} \int |f_n(x)|\, dx = \infty$.

Problem 51

Construct probability measures $\mu_n, \nu_n, n \geq 1$ on $[0, 1]$ such that $\int f d\mu_n - \int f d\nu_n \to 0$ for every continuous function $f : [0, 1] \to \mathbb{R}$ but $\mu_n([0, x]) - \nu_n([0, x]) \nrightarrow 0$ for any $x \in [0, 1)$.

Problem 52

Let $(\Omega, \Im, P)$ be a probability space and $X, X_1, X_2, \ldots$ be random variables on it. Show that $X_n \xrightarrow{P} X$ if and only if $Q \circ X_n^{-1} \xrightarrow{w} Q \circ X^{-1}$ for every probablility measure Q on $(\Omega, \Im)$ which is equivalent to P (in the sense $P << Q$ and $Q << P$)

Problem 53

Let A and B be any two proper subsets of $\mathbb{R}$. Show that $\mathbb{R}^2 \backslash (A \times B)$ is connected.

Suppose $\mathbb{R}^2 \backslash (A \times B) = (A^c \times \mathbb{R}) \cup (\mathbb{R} \times B^c)$ and any two points if this set can be joined by at most three line segments.

Cor: $(\mathbb{Q} \times \mathbb{Q})^c$ is connected in $\mathbb{R}^2$.

Remark: $\mathbb{R}^2$ can be replaced by the product of any two connected spaces.

Problem 54

Find all maps $f : \mathbb{R} \to \mathbb{R}$ such that f is both additive and multiplicative.

Problem 55

What happens if $\mathbb{R}$ is replaced by $\mathbb{C}$ in Problem 55 and f is assumed to be continuous?

Remark: continuity is essential. There exist additive, multiplicative, one-to-one dis-continuous functions on $\mathbb{C}$!.

Problem 56

Let T be a compact operator on a Hilbert space H with orthonormal basis $\{e_1, e_2, ...\}$. Show that $\|Te_n\| \to 0$.

Problem 57

Show that there is a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ converging pointwise which does not converge uniformly on any open interval in $\mathbb{R}$. In contrast, show that if a sequence of analytic functions on a region Ω in $\mathbb{C}$ converges pointwise then there is a non-empty open subset D of Ω such that the sequence converges uniformly on D.

Problem 58

Let μ be a finite positive measure on $(1, \infty)$ and $f(y) = \int_1^\infty \cos(xy) d\mu(x)$. Show that f has at least one zero on $[0, \pi]$.

Problem 59

Consider the following sets of 3×3 real matrices:
a) $\{A : \det(A) = 0\}$
b) $\{A : A \text{ is symmetric}\}$
c) $\{A : A^n = 0 \text{ for some } n \in \mathbb{N}\}$
Treating a 3×3 real matrix as an element of $\mathbb{R}^9$ show that above sets of Lebesgue measure 0.

Problem 60

Let $\{f_n\}$ be an orthonormal set in $L^2([0,1])$ and $A = \{x : \lim\limits_{n\to\infty} f_n(x) \text{ exists}\}$ and let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $x \in A$. Show that $f = 0$ a.e. on A.

Problem 61

Let $T : l^\infty \to \mathbb{R}$ be a linear map such that for any $x = \{x_n\} \in l^\infty$, $T(x) = \lim x_{n_j}$ for some subsequence $\{n_j\}$ of $\{1, 2, ...\}$. Show that T is continuous and multiplicative.

Problem 62

Let $c_1, c_2, ..., c_n$ be distinct complex numbers. Show that $\sum\limits_{k=1}^{n} \prod\limits_{j\neq k} \frac{c_j - c}{c_j - c_k} = 1$ for all $c \in \mathbb{C}$.

Problem 63

Compute $\limsup |a^n - b^n|^{1/n}$ for any two complex numbers a and b.

Problem 64

Prove the identity $[x] + [x + 1/n] + ... + [x + \frac{n-1}{n}] = [nx]$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.

Problem 65

If $f : [0,1] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq C\,|x - y|\ \forall x, y$ prove that given $\epsilon > 0$ there is a polynomial p such that $|p(x) - p(y)| \leq C\,|x - y|\ \forall x, y$ and $|f(x) - p(x)| < \epsilon\ \forall x$.

Problem 66

If $f : [0,\pi] \to \mathbb{R}$ is continuous and $\int\limits_0^\pi f(x) \sin x dx = \int\limits_0^\pi f(x) \cos x dx = 0$ show that f has at least two zeros in $[0,\pi]$.

Problem 67

If $f : \mathbb{R} \to \mathbb{R}$ is non-increasing show that it has a unique fixed point. Use this to show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x$ $\forall x \in \mathbb{R}$.

Remarks: for any n the only continuous function f on $\mathbb{R}$ whose $n-th$ iterate $f_{(n)}$ is the identity function is the identity function itself. The only continuous function f on $\mathbb{R}$ such that $f_{(n)}(x) = -x$ $\forall x$ is $-x$ if n is odd and there is no such function if n is even.

Problem 68

If $f : [0,1] \to \mathbb{R}$ is continuous show that $\frac{1}{n}\sum_{j=1}^{n}(-1)^j f(\frac{j}{n}) \to 0$ as $n \to \infty$.

Problem 69

If n is a positive integer find the precise number of real roots of the equation $\sum_{k=0}^{n} \frac{x^k}{k!}$.

Problem 70 (universal power series)

Show that there is a power series $\sum_{k=1}^{\infty} c_n x^n$ (with no constant term) such that for any continuous function $f : [0,1] \to \mathbb{C}$ with $f(0) = 0$ there is a subsequence $\{s_{n_k}\}$ of the sequence of partial sums of this series converging uniformly to f on $[0,1]$.

Remark: an arbitrary continuous function cannot expressed in the form $\sum_{k=0}^{\infty} c_n x^n$ with the series conveging pointwise. Such a representation would force f to be the restriction to $[0,1)$ of an analytic function on $\{z : |z| < 1\}$.

Problem 71

Show that $\int_{-\infty}^{\infty} (\frac{\sin x}{x})^{2n} \cos(2xy)dx = 0$ if $|y| > 2n$. Also show that the integral is > 0 for all other values of y.

Problem 72

Let $f \in C[0,1]$ and $f(0) = 0$. Show that there is a sequence of polynomials $p_n(x) = \sum\limits_{k=1}^{k_n} a_{k,n} x^k$ converging pointwise to f on $[0,1]$, uniformly on $[\delta, 1]$ $\forall \delta \in (0,1)$, such that $a_{k,n} \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$.

Problem 73

If $f : (0,1) \to (0,\infty)$ is decreasing show that $\dfrac{\int\limits_0^1 x f^2(x)dx}{\int\limits_0^1 x f(x)dx} \leq \dfrac{\int\limits_0^1 f^2(x)dx}{\int\limits_0^1 f(x)dx}$.

Problem 74

If f and g are continuous functions on $(0,1)$ and $g(x) > 0$ $\forall x$ show that $\lim\limits_{n\to\infty} \dfrac{\int\limits_0^1 x^n f(x)dx}{\int\limits_0^1 x^n g(x)dx}$ exists.

Problem 75

Say that two functions $f, g : \mathbb{R} \to \mathbb{R}$ are similar if there is a bijection $\phi : \mathbb{R} \to \mathbb{R}$ such that $f = \phi^{-1} \circ g \circ \phi$. Prove that x^n and x^m are similar if $n = m^k$ for some k (or $n = m^k$ for some k). Are x^2 and $x^2 + 1$ similar? Prove that x^n and x^m are similar if n and m are both odd and greater then 1. Prove that sin and cos are not similar.

Problem 76

Show that there is a sequence of polynimials converging pointwise, but not uniformly, to a *continuous* function on $[0,1]$.

Problem 77

a) Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\{(x,y) : y \neq f(x)\}$ is open the f is continuous.

b) Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\{(x,y) : y > f(x)\}$ and $\{(x,y) : y < f(x)\}$are open the f is continuous.

Problem 78

Let (X, d) be a metric space. Show that X is separable if and only if there is an equivalent metric on it which makes it totally bounded.

Remark: it is clear from above proof that the two equivalent conditions are also equivalent to the existence of a compact metric space Y such that X is homeomorphic to a subset of Y.[In other words, X has a metrizable compactification].

Problem 79

Let $f : \mathbb{R} \to \mathbb{R}$ be additive. Show that the following statements are equivalent:

a) f is continuous

b) $f^{-1}\{0\}$ is closed

c) f is bounded on some open interval containing 0

d) $f(U)$ is not dense in $\mathbb{R}$ for some open set U containing 0

Problem 80

Let (X, d) be a metric space. Consider the following properties of X :
a) Every real continuous function on X is bounded
b) Every real continuous bounded function on X attains its supremum
c) Every real continuous function on X is uniformly continuous
d) The image of every real continuous function on X is connected
e) $d(A, B) > 0$ whenever A and B are disjoint closed sets in X
Do any of the first the conditions a),b),c),e) imply that X is compact? Does d) imply that X is connected?

Problem 81

a) Suppose $f : \mathbb{R} \to \mathbb{R}$ has a left limit $f(x-)$ at every point and suppose $f(x-)$ is continuous at a. Does it follow that f is continuous at a? What if $f(x-) \to f(a)$ as $x \to a$?

b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a left derivative $f'(x-)$ at every point. Suppose $f'(x-)$ is continuous at a. Show that f is differentiable at a.

Problem 82

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has a local minimum at each point. Show that its range is atmost countable. Construct an example of such a function which is increasing and which has the properties $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$. If f has a local minimum at each point and if f is also continuous show that it is a constant.

Problem 83

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$. If f is a non-constant convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x)\ \forall x$ show that it is identity on $[a, \infty)$ for some real number a and give an example of such a function. Prove

that there is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ other then the identity such that $f(f(x)) = f(x)\ \forall x$.

Problem 84

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and $f : X \to Y$. Prove or disprove the following:

a) if $(f^{-1}(A))^0 \neq \emptyset$ whenever $A^0 \neq \emptyset$ then f is continuous

b) if $X_1 = X_2 = X$ (say) and a set A is sense in X w.r.t. τ_1 if and only if it is dense in X w.r.t. τ_2 then $\tau_1 = \tau_2$.

c) if $(f(A))^0 \neq \emptyset$ whenever $A^0 \neq \emptyset$ then f is an open map

Problem 85

Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that the smallest topology that makes f continuous (w.r.t the usual topology on the range) is the power set of $\mathbb{R}$?

Problem 86

Prove that a function f from one metric space to another is uniformly continuous if and only if $d(A, B) = 0$ implies $d(f(A), f(B)) = 0$. [$d(A, B)$ is the distance between the sets A, B].

Problem 87

An additive subgroup of $\mathbb{R}$ is either dense or discrete. There are additive subgroups which are dense and of first category and there are subgroups second category as well.

Problem 88

Characterize metric spaces (X, d) such that pointwise convergence of a sequence real continuous functions on X implies uniform convergence.

Problem 89

Let $f : \mathbb{R} \to \mathbb{R}$ map intervals to intervals. Does it follow that f is continuous? What if f is also one-to-one?

Remark: If $f^{-1}(\{a\})$ is empty or a finite set for each a and if f has intermediate value property then it is continuous.

Problem 90

Let $f : (0, \infty) \to (0, \infty)$ be a convex function and $a, b \in \mathbb{R}$. Show that $xf(a + \frac{b}{x})$ is a convex function on $(0, \infty)$.

Problem 91

Let A, B, C be subsets of a normed linear space X such that $A + C \subset B + C$ and C is bounded. Show that A is contained in the closed concex hull of B

Problem 92

Let A, B, C, D be $n \times n$ matricies such that $AD^* - BC^* = I, AB^* = BA^*$ and $CD^* = DC^*$. Prove that $A^*D - C^*B = I$.

Problem 93

Let (X, d) be a metric space such that for any $x_1, x_2 \in X$ there exists $u \in X$ with $d^2(x_1, x_2) + 4d^2(x, u) \leq 2d^2(x_1, x) + 2d^2(x_2, x)$ for all $x \in X$ show that u is uniquely determined by x_1 and x_2 and that $d(u, x_1) = d(u, x_2) = \frac{1}{2}d(x_1, x_2)$. Prove or disprove that $d^2(x_1, x_2) + 4d^2(x, \frac{x_1+x_2}{2}) \leq 2d^2(x_1, x) + 2d^2(x_2, x)$ for all $x \in X$ when X is a normed linear space.

Problem 94

True or false: if X is a normed linear space then $\|x - y\|^2 + \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \ \forall x, y \in X$.

True or false: if X is a normed linear space then $\|x - y\|^2 + \|x + y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 \ \forall x, y \in X$.

Problem 95

Let X be a normed linear space and $f : X \to \mathbb{R}$ is *locally convex* in the sense for each $x \in X$ there exists $\delta > 0$ such that f is convex on $B(x, \delta)$. Does it follow that f is convex on X?

Problem 96

Let $\{\phi_n\}$ be a sequence of continuous functions : $(0, \infty) \to (0, \infty)$. Show that there is a continuous function $f : (0, \infty) \to (0, \infty)$ which $\to \infty$ faster then each of the $\phi_n' s$ [i.e. $\lim\limits_{x \to \infty} \frac{f(x)}{\phi_n(x)} = \infty$ for each n]

Remark: There is an entire function g such that $g(x) \geq \phi_n(x)$ $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}$. In particular g is "smooth".

Problem 97

Prove that the Hausdorff dimension of the Cantor's ternary set C is $\frac{\log 2}{\log 3}$

Problem 98

Show that there is no sequence $\{a_n\}$ converging to 0 such that $\hat{f}(n) \to 0$ faster then $\{a_n\}$ for every continuous function f on $\mathbb{R}$ with period 2π.["$\hat{f}(n) \to 0$ faster then $\{a_n\}$" means $\frac{\hat{f}(n)}{a_n} \to 0$].

Problem 99

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim\limits_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = 0$ $\forall x$. Prove that $\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = 0$ for all x and all $h \in \mathbb{R}$. Find all functions f with this property.

Problem 100

Let $\{a_n\}$ be a sequence of real numbers such that $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for all $x > 0$. Show that the equation $\int\limits_0^{\infty} e^{-x} \sum\limits_{n=0}^{\infty} a_n x^n dx = \sum\limits_{n=0}^{\infty} a_n \int\limits_0^{\infty} e^{-x} x^n dx$ holds if the series on the right is convergent.

Remark: this is a result on interchange of limit and integral where the basic theorems of measure theory don't seem to be of much use!

Problem 101

If the graph of $f : \mathbb{R} \to \mathbb{R}$ is closed and connected then f is continuous. This does not extend to maps between general connected metric spaces.

Problem 102.

Let $I = (a, b)$ be a finite or infinite open interval in $\mathbb{R}$ and d be a metric on it which is equivalent to the usual metric. Prove that there exist disjoint closed sets A and B in I such that $d(A, B) = 0$.

Problem 103

Suppose $A \subset \mathbb{R}^n$ is such that the distance between any two points is rational. Prove that A is atmost countable.

Problem 104

Let $A \subset \mathbb{R}^n$ be countable. Show that $\mathbb{R}^n \backslash A$ is connected.

Problem 105

Let X be a separable normed linear space and f be a continuous linear functional on a subspace M of X. Show without using Zorn's Lemma (or any of its equivalents) that f can be extended to a continuous linear functional on X with the same norm.

Problem 106

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) > \int\limits_0^x f(t)dt$ $\forall t \in [0,1]$. Prove that $f(x) > 0 \ \forall x \in [0,1]$.

Is the following discrete analog true?

If $a_1, a_2, ..., a_N$ are real numbers such that $a_{k+1} > a_1 + a_2 + ... + a_k$ for $1 \leq k < n$ then $a_k > 0$ for all k.

Problem 107

Let $p(x) = x^2 + ax + b$ and A be the 3×3 matrix with entries $p(i-j), 0 \leq i, j \leq 2$. Show that the determinant of A does not depend on the coefficients of p.

Remarks: the argument actually works for monic polynomials of any degree and the value of the determinant is $(n!)^{\ n+1}$ when $p(x)$ is of the type $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$.

Problem 108

Let A be a bounded set in a Hilbert space. Show that there is a unique closed ball of minimal radius containing A.

Problem 109 [See also Problem 1]

Let μ be a finite positive measure on the Borel subsets of $(0, \infty)$. If $g \in L^\infty(\mu)$ and $\int\limits_0^\infty e^{-x} p(x) g(x) d\mu(x) = 0$ for every polynomial p show that $g = 0$ a.e. $[\mu]$. Conclude that $\{e^{-x}p(x) : p \text{ is a polynomial}\}$ is dense in $L^1(\mu)$.

Problem 110

Find all continuous functions $f:(0,\infty)\to(0,\infty)$ such that $x\to\int\limits_x^{kx} f(t)dt$ is constant on $(0,\infty)$.

Problem 111

Let $A\subset\mathbb{C}$ be a convex set such that $x\in A\Rightarrow -x\in A$. If $a_1,a_2,a_3\in A$ show that at least one of the 6 numbers $a_1+a_2, a_1-a_2, a_2+a_3, a_2-a_3, a_3+a_1, a_3-a_1$ must be in A.

Problem 112

Show that every polynomial p with real coefficients and real roots satisfies the inequality $(n-1)[p'(x)]^2\geq np(x)[p''(x)]$ where n is the degree of p.

Problem 113

Find $\sup\{\frac{(\int\limits_0^1 f(x)dx)^2(\int\limits_0^1 g(x)dx)^2}{\int\limits_0^1 [f(x)]^2dx\int\limits_0^1 [g(x)]^2dx} : f,g:[01]\to\mathbb{R}$ are continuous, $\int\limits_0^1 f(x)g(x)dx = 0\}$.

Problem 114

a) Let U be an open set in $\mathbb{R}, F$ a closed set and $U\subset F$. Show that there is a set A whose interior is U and closure is F.

b) Find all sets $A\subset\mathbb{R}$ such that $A=\partial B$ for some $B\subset\mathbb{R}$.

Remark: only two properties of Q are required: it has no interior and it is dense. The results therefore extend to any topological space in which such a set exists. [Countability of Q is not required]

Problem 115

Let H be a Hilbert space and C be a closed convex subset. For any $x\in H$ let Px be the unique point of C that is closest to C. Show that $\|x-y\|^2\geq\|x-Px\|^2+\|y-Px\|^2\ \forall y\in C$.

Remark: The definition only says $\|x-y\|^2 \geq \|x-Px\|^2 \ \forall y \in C$. It is interesting to note that there is always a lower bound for the difference $\|x-y\|^2 - \|x-Px\|^2$.

Problem 116

Let $\{x \in \mathbb{R}^n : \|x\| = 1\} \subset \bigcup_{j=1}^{n} \bar{B}(x_j, r_j)$ where $\bar{B}(x_j, r_j)$ is the closed ball with center x_j and radius r_j. Show that $0 \in \bar{B}(x_j, r_j)$ for some j. Show that the conclusion is false if the number of closed balls is allowed to exceed n.

Problem 117

Let C be a closed convex set in a Hilbert space H. Let $P(x)$ be the point of C closest to x. Show that $\|P(x) - P(y)\| \leq \|x - y\| \ \forall x, y \in H$. [See also Problem 118 below].

Problem 118

In Problem 117 show that $\|P(x) - P(y)\| < \|x - y\|$ unless $P(x) - P(y) = x - y$.

Problem 119

Let $f : [0, \infty) \to [0, \infty)$ be non-decreasing with $\int_1^\infty \frac{1}{f(x)} dx = \infty$. Show that $\int_1^\infty \frac{1}{x \log(f(x))} dx = \infty$. Can we also assert that $\int_1^\infty \frac{1}{x \log(f(x)) \log(\log(f(x)))} dx = \infty$?

Problem 120

a) Let (X, d) be a compact metric space and $T : X \to X$ be onto. If $d(Tx, Ty) \leq d(x, y) \ \forall x, y$ prove that $d(Tx, Ty) = d(x, y) \ \forall x, y$.

b) Let (X, d) be a compact metric space and a continuous map $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y) \ \forall x, y$. Prove that the conclusion of part a) holds.

Remark: several improvements of these results are given in the next few problems.

Problem 121

Let (X, d) be a compact metric space and $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y)$ for all $x, y \in X$. Then T is an isometry of X onto itself. [Thus continuity of T need not be assumed in previous problem]

Problem 122

Find an error in the following proof given in Amercan Math. Monthly, vol. 98, no. 7, 1991 (p. 664).

Let X be a compact metric space and $T : X \to X$ be any map with $\inf\limits_{n\geq 1} d(T^n x, T^n y) > 0$ whenever $x \neq y$. Show that $T(X) = X$. Solution: let $D(x,y) = \inf\limits_{n\geq 0} d(T^n x, T^n y)$ where $T^0 = I$. D is a metric and $D \leq d$. It follows by compactness of X that the identity map $i : (X,d) \to (X,D)$ is a homeomorphism and (X,D) is a compact metric space. By definition $D(Tx,Ty) \geq D(x,y)$. By Problem 121 above T is an isometry of X onto itself.

Problem 123

Is the product of two derivatives on $\mathbb{R}$ necessarily a derivative?

Problem124

Let $p, q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and f, g be non-negative continuous functions on $\mathbb{R}$ with compact support. Show that $\displaystyle\int \sup_y \{f(x-y)g(y)\}dx \geq \|f\|_p \|g\|_q$.

Problem 125

a) Find all positive numbers α such that there is a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq a[f(x)]^\alpha$ for all x sufficiently large for some $a \in (0,\infty)$.

b) Does there exist a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq af(f(x))$ for all x sufficiently large for some $a \in (0,\infty)$?

Problem 126

Let $a_n > 0$ and $\displaystyle\sum_{n=1}^{\infty} a_n \log(1 + \frac{1}{a_n}) < \infty$. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{\|x-b_n\|^k} < \infty$ almost everywhere for any sequence $\{b_n\} \subset \mathbb{R}^k$. [$\|\|$ is the norm in $\mathbb{R}^k$].

Problem 127

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f \circ g$ is Riemann integrable on $[0,1]$ whenever $g : [0,1] \to \mathbb{R}$ is continuous. Show that f is continuous on $\mathbb{R}$.

Problem 128

Is the set of all $n \times n$ invertible matrices dense in the space of all $n \times n$ matrices? Is the space of all invertible operators on a Hilbert space dense in the space of all operators on that space?

Problem 129

Let A be any $n \times n$ matrix. For any positive integer k Show that there is a unique $n \times n$ matrix B such that $B(B^*B)^k = A$.

Probem 130

Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous at 0 if and only if $f(x_n) \to 0$ wheneve $x_n \to 0$. Can differentiability of f be characterized by the condition $\sum f(x_n)$ converges whenever $\sum x_n$ converges?

Problem 131

Find a necessary and sufficient condition for f to be CP. [See Problem 130 for definition of CP].

Problem 132

Let $f : (0,1) \to (0,1)$ be a continuous function such that for any $x \in (0,1)$ there is an integer n such that $f_{(n)}(x) = x$ where $f_{(1)} = f$ and $f_{(n)} = f \circ f_{(n-1)}$ for $n \geq 2$. Show that $f(x) = x \ \forall x \in (0,1)$. Is the result true of $(0,1)$ is replaced by $[0,1]$?

Problem 133

a) Let $f : \{0,1,2...\} \to \{0,1,2...\}$ satisfy $f(m^2+n^2) = f^2(m) + f^2(n)$ $\forall m,n \geq 0$. Show that either $f(n) = 0$ for all n or $f(n) = n$ for all n

b) Let $f : [0,\infty) \to [0,\infty)$ satisfy $f(x^2+y^2) = f^2(x) + f^2(y) \ \forall x,y \geq 0$. If f is continuous show that $f \equiv 0$ or $f \equiv \frac{1}{2}$ or $f(x) = x$ for all x.

Problem 134

Let C be a bounded subset of $V \equiv \mathbb{R}^n$ or $\mathbb{C}^n$ such that for each $x \in V$ there is a unique point Px of C which is closest to it. Show that C is closed and convex.